Corrected Momentum and Energy Equations Disprove Betz's Limit

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To this day, much of modern wind turbine design and optimization is based on actuator disc theory which uses an incorrect solution to the momentum equation to derive results that diverge from realistic airflow. This along with the incorrect use of Bernoulli's equation to equate the pressure differential across the turbine to the loss in kinetic energy results in the implication of false limitations on the maximum theoretical power extraction from a wind turbine. This paper derives new solutions for the momentum and energy equations which align with natural observations and empirical data and mathematically disprove Betz's Limit. A thermodynamic model including rotation of the wind turbine wake is developed based on equations for isentropic flow. The key to extracting more energy from the wind is shown to be accomplished by increasing the rotational parameters of the slipstream with the naturally occurring extraction of thermal energy.

Nomenclature

A	=	area
a	=	axial induction factor, $a = (V_1\text{-}V_2)/ V_1$
a_i	=	inflow velocity ratio, $a_i = V_2/ V_1 = (1\text{-}a)$
B	=	number of blades
b	=	axial slipstream factor, $b = (V_1\text{-}V_6)/ V_1$
b_i	=	outflow velocity ratio, $b_i = V_6/V_1 = (1\text{-}b)$
c_p	=	specific heat at constant pressure
c_v	=	specific heat at constant volume
e	=	energy per unit mass
h	=	enthalpy per unit mass
K	=	ratio of change in enthalpy to ke
k	=	specific heat ratio c_p/c_v
F	=	force
ke	=	kinetic energy per unit mass
M	=	momentum
$\dot{m}$	=	mass flow $\dot{m} = \rho VA$
P	=	power
p	=	pressure
q	=	dynamic pressure $q=\rho V^2/2$
R	=	maximum radius or gas constant in equation of state
r	=	local or relative radius of blade element
T	=	thrust or temperature
V	=	velocity, with subscript defining location
λ	=	tip speed ratio, $\lambda = \Omega R/V_1$, λ_R may also be used
λ_r	=	local speed ratio, $\lambda_r = \Omega r/V_1$
λ_s	=	slipstream speed ratio, $\lambda_s = \omega r/V_1$
λ_S	=	slipstream outer speed ratio , $\lambda_S = \omega R/V_1$
ρ	=	density
τ	=	torque
Ω	=	angular velocity of turbine
ω	=	angular velocity of slipstream

Subscripts

e	=	exit or energy per unit mass
i	=	initial or inlet
n	=	normal to turbine disc.
r	=	with respect to an annular element located at radius r
θ	=	with respect to the direction of rotation, tangentially

I. Introduction

Modern wind turbine output has been increasing with the use of faster turning, larger diameter rotors on higher towers, all pushing the limits of practical manufacturing, transportation and construction. At the same time the potential of smaller distributed energy sources beckons for new and better solutions. But in almost a century, little has changed within the Blade Element Momentum Theory used to design and optimize wind turbine configurations. This is due in part to assumptions, mistakes and misconceptions rooted in propeller and rotorcraft theory that have become ingrained into the historical basis of wind turbine design. To this day, Froude's actuator disc concept with an incorrect solution to the momentum equation and misuse of Bernoulli's equation is used to derive invalid limitations on wind turbine output. The goal of this paper is to challenge these current theories and correct the long held beliefs that are stifling innovation.

This paper is written with the assumption that the readers are familiar with conventional wind turbine design theories and basic blade element analysis, if not I refer you to the excellent references[1,2,3,4] listed at the end of this paper. Although it seems redundant to reproduce here many of the details of conventional theory as is so often done, I must do so in order to point out and correct the errors in the underlying assumptions and misconceptions.

II. Misuse of 1-D Momentum and Bernoulli's Equations

I begin by redefining some terms and station positions for identifying the flow variables in the mathematical model. This numbering system varies from conventional numbering in order to more precisely describe the wake and regions near the turbine. Referring to Fig. 1 the airflow through the turbine is commonly represented as a stream tube. At this point we will make no commitment to the shape of this stream tube either upstream or downstream, understanding that this shape will be a result of the turbine design and conditions of operation. Station 1 represents the initial position of influence and station 6 the final position of influence. Stations 2 and 3 represent positions just forward and aft of the turbine respectively. Stations 4 and 5 are reserved for later discussion. Subscripts 0 and ∞ will be reserved for total and free stream conditions respectfully with $V_1 = V_\infty$.

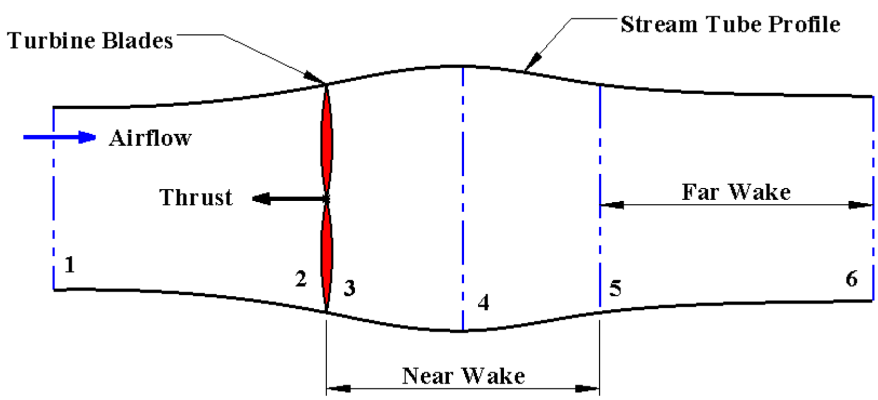

Figure 1 Station Positions

Convention defines axial induction factor ***a*** as the ratio of induced flow velocity, the decrease in inflow velocity, over the free stream velocity or $a = (V_1\text{-}V_2)/\ V_1$. In order to simplify future equations I will define a more convenient term, the inflow velocity ratio as $a_i = V_2/\ V_1 = (1\text{-}a)$. Glauert[1] also uses a term called the axial slipstream factor which for a wind turbine would be defined as $b = (V_1\text{-}\ V_6)/\ V_1$. I will similarly define outflow velocity ratio as $b_i = V_6 / V_1 = (1\text{-}b)$.

Most conventional theories and versions of Blade Element Momentum method, BEM are in some manner based on the misuse of both simple one dimensional momentum theory and Bernoulli's equation. The underlying and questionable assumptions for 1-D Momentum Theory are an ideal wind turbine modeled by a semi-permeable disc being acted on by a one dimensional ideal fluid flow which is inviscid (frictionless), incompressible and with no wake rotation. If there is no wake rotation there is no power being extracted from the conventional turbine. Furthermore 1-D momentum theory is usually used to evaluate internal pipe flow and is not necessarily relevant to free stream air flow as we shall see. But ignoring all this, the theory is still important in predicting the thrust and axial induction factor of the turbine if we properly correct for these poor assumptions.

In its simplest form the momentum equation follows from Newton's 2nd law that the sum of the forces acting on a body is equal to its change in momentum.

$$\sum F = \frac{d(mV)}{dt} = \dot{m}(V_{out} - V_{in}) \tag{1}$$

In this case we are dealing not with a body but a control volume of air. Conventional theory uses one of two control volumes, these are shown here in Fig. 2 and Fig. 3 using the present nomenclature and station positions.

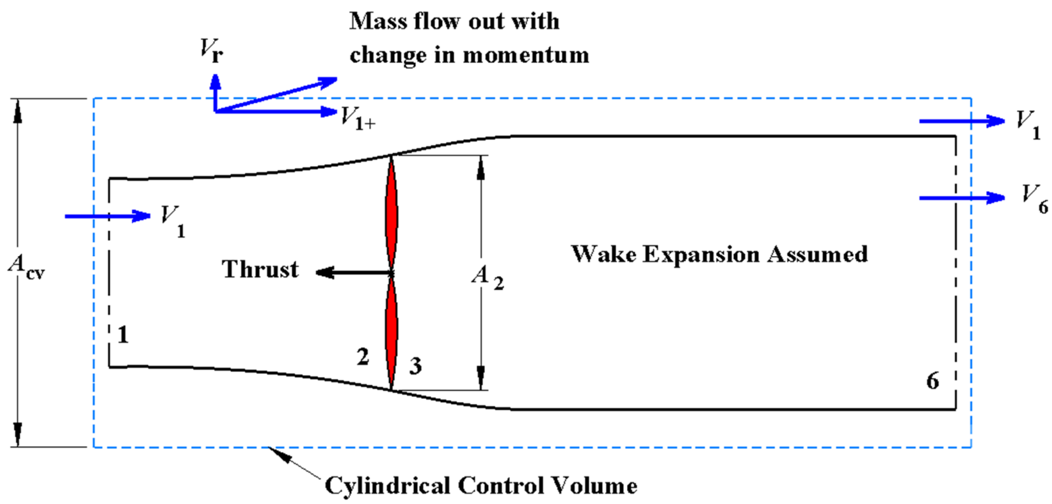

Figure 2 Conventional CV I

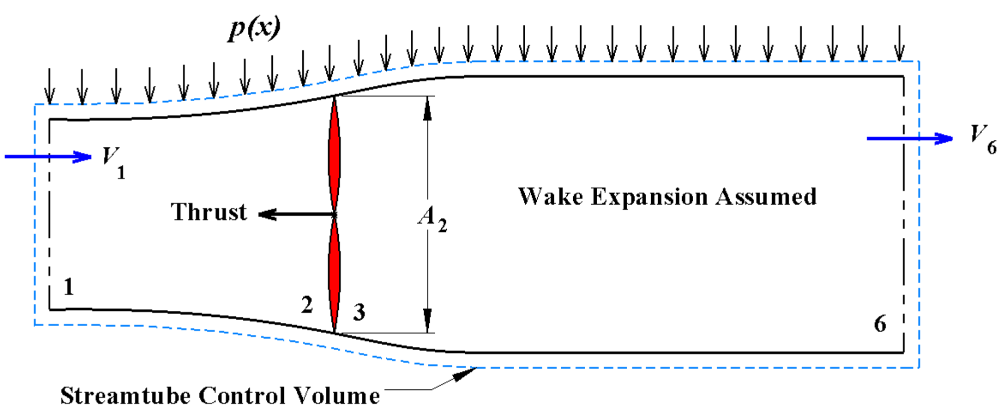

Figure 3 Conventional CV II

The first control volume diagram Fig. 2 above, accounts for the flow both through the turbine and at some dimension external to it parallel with the freestream airflow. Therefore as the airflow is deflected around the turbine some mass flow and momentum must exit out the side of the control volume, which immediately contradicts the assumption of 1-D flow. In the first derivation, the control volume diagram is given the following equation.

$$\rho V_6^2 A_6 + \rho V_1^2 (A_{cv} - A_6) + \dot{m}_{side} V_1 - \rho V_1^2 A_{cv} = -T \quad (2)$$

The change in momentum out the side is assumed to be its mass flow times the velocity of the air stream V_1. The conservation of mass is used to calculate the mass flow out the side as

$$\dot{m}_{side} = \rho A_6 (V_1 - V_6) \quad (3)$$

From here the thrust or drag on the turbine becomes

$$T = \rho V_6 A_6 (V_1 - V_6) = \dot{m}(V_1 - V_6) \quad (4)$$

The error in this solution is assuming the change in momentum out the side to be equal to its mass flow times the velocity of the air stream V_1. If the airflow is exiting the control volume as it approaches the turbine at varying stream tube diameters then at each elemental station a portion will exit at a different velocity. In this case a different speed as well as direction or more correctly with a radial component, term V_r shown on Fig.2. This causes a change in momentum which when combined with shear forces in the velocity gradient must balance with the changing static pressure at each elemental x station. The assumption that the change in momentum out the side is equal to its mass flow times the velocity of the air stream V_1 is valid only if the x component of the flow is equal to V_1 and the wake only expands and does not contract down wind. We know in free flow around a sphere the air at the perpendicular circumference will accelerate to $1.5V_1$ so there is no reason to believe $V_x = V_1$. We also know from observing flow around normal obstructions that the fluid will flow around the obstacle and then back in behind it, so the assumption of wake expansion at station 6 could also be challenged which would invalidate the above solution.

In the alternative control volume Fig. 3 above, the flow is strictly considered only within the streamlines which pass through the turbine. The sum of forces must be integrated along the pressure distribution of the outer streamline surface. This corrected control volume diagram yields

$$T = \dot{m}(V_1 - V_6) - F_{pressure} \quad (5)$$

This equation is correct. But conventional theory compares Eq. (4) to Eq. (5) and concludes that $F_{pressure}$ must equal zero. The correct conclusion should have been that the derivation of Eq. (4) is oversimplified and incorrect for the reasons mentioned. Again, the airflow in Fig.2 exits the control volume at differing radial velocities as it approaches the turbine creating an external pressure distribution. This external pressure distribution along with the external shear forces must balance with varying static pressures within the enclosed streamlines of Fig.3. The integration of these pressures will not be equal to zero but will equal some positive value resulting from a complicated nonlinear relationship acting on the control volume along the perimeter of the outer streamline. These calculations are currently beyond myself and this paper, although they should be considered significant enough to discount the validity of basic 1-D momentum theory.

The next step in conventional theory is to reconcile the results from this momentum equation with Bernoulli's Equation. A further foundation for conventional theory is the misuse of Bernoulli's equation to relate the pressure drop across the turbine to the energy extracted and change in kinetic energy far down stream. Even though Bernoulli's equation is invalid across any device which changes the total energy of the flow, conventional theory incorrectly carries this out by comparing the equations up steam and down steam of the turbine as follows

$$\textit{Up Stream} \quad p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 \quad (6)$$

$$\textit{Down Stream} \quad p_3 + \frac{1}{2}\rho V_3^2 = p_6 + \frac{1}{2}\rho V_6^2 \quad (7)$$

But these equations are invalid as written. We defined stations 1 and 6 as the limits of the turbines influence. The flow in these regions is not an internal pipe flow but is the result of the turbines influence and therefore Bernoulli's Equation does not apply.

Conventional theory carries on regardless with Bernoulli's Equation, assuming that the airflow is a constant density or incompressible and at constant temperature. Therefore since mass flow must be conserved across the turbine $V_3 = V_2$. If Δp is defined as $\Delta p = p_3 - p_2$ then $p_3 = p_2 + \Delta p$. Furthermore it is assumed the final pressure far down stream equals the original static pressure $p_6 = p_\infty$. Eq. (7) then becomes

$$p_2 + \Delta p + \frac{1}{2}\rho V_2^2 = p_\infty + \frac{1}{2}\rho V_6^2 \quad (8)$$

Solving Eq. (6) and Eq. (8) for Δp yields

$$\Delta p = \frac{1}{2}\rho\left(V_6^2 - V_1^2\right) \quad (9)$$

This result, Eq. (9) is an incorrect solution in contradiction to the energy equation. The energy extraction occurs between station 2 and 3, and pressure cannot change without a change in temperature which negates Bernoulli's premise. We will come back to and elaborate on this later.

The thrust or drag on the turbine is then assumed to be equal to this pressure drop times the area of the turbine and this is equated to Eq. (4).

$$\frac{1}{2}\rho\left(V_1^2 - V_6^2\right) A_2 = \rho V_2 A_2 \left(V_1 - V_6\right) \quad (10)$$

Solving Eq. (10) for V_2 yields

$$V_2 = \frac{1}{2}\left(V_1 + V_6\right) \quad (11)$$

or in terms of velocity ratios

$$a_i = \frac{1}{2}\left(b_i + 1\right) \quad (12)$$

This is a fundamental and incorrect premise of conventional theory that the velocity through the turbine is equal to the average of the free stream and final velocity of the affected airflow. This is derived based on ignoring slipstream rotation and in conjunction with misuse of the linear momentum and Bernoulli's equations.

Continuing with the conventional solution, solving for b_i yields

$$b_i = 2\,a_i - 1 \quad (13)$$

Conventional theory in Eq. (13) implies that if the inflow velocity ratio is one half of the free stream velocity then the final velocity will equal zero. This by conservation of mass flow implies an expansion of the wake to an infinite diameter at an inflow velocity ratio of 0.5, this would be a physical impossibility and is where the conventional solution to the momentum theory diverges from reality. This is an accepted limitation of the 1-D Momentum Theory that it is invalid as it approaches axial induction factors of 0.5. It is often said to be associated with the condition of a turbulent wake state. But within this derivation there is no condition predicting turbulent versus laminar flow around the turbine. There is also no reason for one to accept this solution as being anything but invalid over a greater range since we will see it does not resemble the empirical data.

From here conventional theory calculates thrust and power coefficients based on the previous errors. Equation (11) is used to calculate V_6 in terms of axial induction factor yielding

$$V_6 = V_1(1 - 2a) \quad (14)$$

Thrust is calculated by entering Eq. (14) into Eq. (4) yielding

$$T = 2\rho a(1-a)V_1^2 A_2 \tag{15}$$

By definition thrust coefficient is calculated by dividing through by dynamic pressure and area, $\frac{1}{2}\rho V_1^2 A_2$, yielding

$$C_T = 4a(1-a) \tag{16}$$

This relationship is graphed in Fig. 9. Notice that this result peaks at a = 0.5 and goes to zero as the axial induction factor goes to 1.0. This likewise does not occur in any empirical data.

III. Betz Debunked

Power in conventional theory is calculated either by the mass flow times the deficit in kinetic energy or from thrust times velocity, both conveniently resulting in a common solution when it is assumed that the pressure drop is related to the change in *ke*. Omitting the derivation and again using axial induction factor as the parameter the result is

$$P = 2\rho V_1^3 A_2 a(1-a)^2 \tag{17}$$

Calculating power coefficient by definition dividing through by $\frac{1}{2}\rho V_1^3 A_2$ leaves

$$Cp = 4a(1-a)^2 \tag{18}$$

Taking the derivative of this and setting equal to zero gives the result known as Betz's limit which claims the maximum C_p attainable is 16/27 at an axial induction factor of 1/3.

The next assumption that I believe was in error here is that power calculations are a function of $F_n V$, thrust times velocity at the turbine disc. This is not a propeller and even if it was there is no direct connection mechanically or mathematically between this term and the power extracted term, $P = \tau \cdot \Omega$. The term $F_n V$ is the power which the turbine is causing to be transferred from kinetic energy to internal energy. The effect of thrust from the wind turbine is to reduce the momentum and increase the internal energy of the airflow; it is not directly tied to the energy extraction. I will cover this in depth when we get to the energy equation.

In summary, as shown, the previously accepted results are based on an incorrect solution to the momentum equation and misuse of Bernoulli's equation to solve an energy problem. The results are neither a law nor a valid limit and serve only to stifle innovation. In addition power extraction is not a result from the reduction in kinetic energy. If the conservation of mass flow requires that the velocity of airflow entering the turbine actuator disc equals the velocity exiting the turbine then the conventional explanation that the wind turbine extracts power from kinetic energy of the wind by slowing it down is incorrect. The energy transfer occurs at the turbine with an assumed negligible change in velocity. Therefore in fact kinetic energy at this point is the medium through which the energy is transferred, but the power in a rotating wind turbine is extracted from the differential pressure across the airfoil which is being converted to negative thrust and torque, but only the torque results in energy extraction. The reduction in kinetic energy downstream is a secondary effect of the natural flow balancing itself with the surroundings. The relevant energy equations must include the rotational terms and are invalid if temperature is ignored because the makeup of the internal energy of the flow is changing.

IV. Corrected Momentum Equation

So what does the corrected momentum equation look like? We must first start with an accurate control volume diagram that works regardless of any assumptions about wake expansion. Let's start with a control volume Fig. 4 that assumes the streamlines flowing around the turbine and coming back together behind it. This is what we know usually occurs in natural disturbances to airflow.

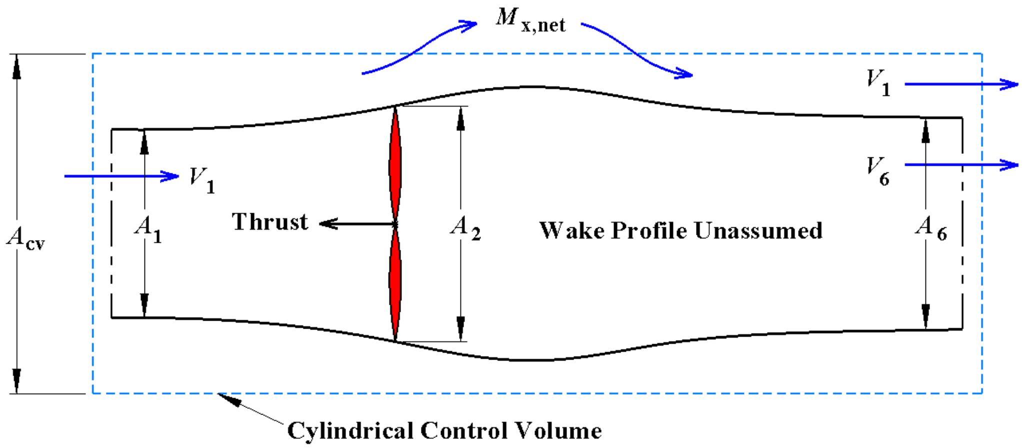

Figure 4 MBET Momentum CV.

In control volume diagram Fig. 4, I make no assumptions about the velocity or mass flow representing the change in momentum out the sides. At this point we will just call it $M_{x,net}$. Summing the forces and equating to the change in momentum through the control volume gives us the following

$$-T = -(\rho V_1 A_{cv})V_1 + M_{x,net} + (\rho V_6 A_6)V_6 + \rho V_1^2 (A_{cv} - A_6) \tag{19}$$

Simplifying we arrive at the corrected momentum equation of

$$T = \rho A_6 \left(V_1^2 - V_6^2\right) - M_{x,net} \tag{20}$$

Although we don't have a direct analytical solution for $M_{x,net}$ we can arrive at an estimated solution by deductive analysis. Examining the limits of the equation, we know that when $V_1 = V_6$ the flow through the turbine must be unrestricted with zero change in momentum, zero thrust and therefore $M_{x,net(V_1=V_6)} = 0$. When $V_6 = 0$ then we have reached a state of total restriction through the turbine so all the mass flow and momentum must be flowing out around the front of turbine and back in behind the turbine. So what is the limit of the thrust and $M_{x,net(V_6=0)}$ for this condition? For turbulent wake states we can consider the turbine drag may be similar to the drag on a flat plate or disc. This is usually referred to in a non-dimension form as coefficient of drag C_D or wind turbine convention likes to refer to it as coefficient of thrust C_T either way

$$C_T = C_D = \frac{(T\,or\,D)}{qA_2} \quad \text{where} \quad q = \frac{1}{2}\rho V_1^2 \tag{21}$$

Depending on the Reynolds number this value can vary from 1.2 to 1.8 for a turbulent wake. But for our limit we want to imagine maintaining the flow as laminar around the turbine. In this case the pressure on the front face of the turbine would be $+q$ and on the back side $-q$ for a total drag force of $2qA_2$ or $C_T = 2$. So for $V_6 = 0$ and $T = 2qA_2 = \rho V_1^2 A_2$ the momentum equation becomes

$$\rho V_1^2 A_2 = \rho A_6 V_1^2 - M_{x,net(V_6=0)} \tag{22}$$

Solving for $M_{x,net(V_6=0)}$

$$M_{x,net(V_6=0)} = \rho V_1^2 (A_6 - A_2) \tag{23}$$

If $V_6 = 0$, then obviously $A_6 = 0$ not ∞ and this equation gives us the result we expect. $M_{x,net(V_6=0)} = \rho V_1^2 A_2 = 2qA_2$.
Using the continuity equation (conservation of mass flow) $V_2 A_2 = V_6 A_6$ we can manipulate Eq. (23) to a final form of

$$M_{x,net(V_6=0)} = 2qA_2\left(\frac{V_2}{V_6} - 1\right) \tag{24}$$

Looking back at our previous limit $M_{x,net(V_1=V_6)}=0$, the continuity equation also implies if there is no restriction to airflow then $V_1=V_2=V_6$. Inserting this into Eq. (24) coincidentally also equals zero. This might imply that Eq. (24) is a general equation approximated by the linear relationship shown in Fig. 5.

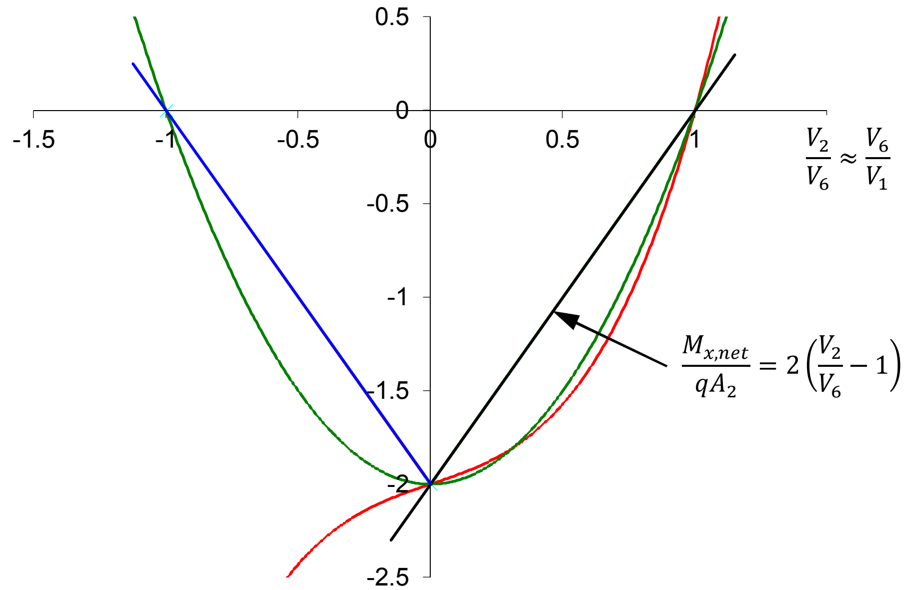

Figure 5 Possible $M_{x,net}$ Solutions

In Fig. 5 various possible solutions are plotted. Increasing negative values of V_2/V_6 and $M_{x,net}$ would be produced by a propeller brake state with energy being added to reversing the airflow through the turbine. The limit for the wind turbine state would be $V_2/V_6=0$. Reversed flow caused by negative values of V_6 would produce the blue negative sloping line. This condition could only be produced in an open throat wind tunnel with flow coming from both directions, with mass flow and momentum flowing out of the control volume parallel to the turbine disc and when $V_6=-V_2=-V_1$ then $M_{x,net}=0$. The intersection of the lines would represent the condition where $A_1=A_6=0$, with no flow through the turbine and a theoretical laminar wake. Possible quadratic or cubic solutions are additionally shown in green and red.

What if we now look at $M_{x,net}$ as a function of the non-dimensional $V_6/V_1= b_i$. We know that if there is no restriction $V_1=V_2=V_6$ then $b_i= 1$ and $M_{x,net}=0$. We also assume that if $V_6=0$ then $b_i= 0$ and $M_{x,net}=-2qA_2$. Let's further assume another point that if $V_6=-1$ which means $b_i=-1$ implying reverse flow in the wake as above, then again we have $M_{x,net}=0$. These points all match the points plotted in Fig. (5) for V_2/V_6.

Admittedly we do not know that these are linear relationships, they could be quadratic, cubic or may be other differing relationships. But we do know based on the corrected momentum equation and within the range of the wind turbines flow parameters the relationship exists that

$$M_{x,net} \approx 2qA_2\left(\frac{V_2}{V_6}-1\right) \approx 2qA_2\left(\frac{V_6}{V_1}-1\right) \tag{25}$$

and observing from this

$$\frac{V_2}{V_6} \approx \frac{V_6}{V_1} \tag{26}$$

Rearranging terms yields

$$\frac{V_2}{V_1} = \left(\frac{V_6}{V_1}\right)^2 \quad \text{or} \quad a_i = b_i^2 \tag{27}$$

or more useful

$$b_i = \sqrt{a_i} \tag{28}$$

Eq. (28) has profound implications for both the momentum and energy equations, changing the way we analyze the airflow in the wake of a wind turbine. Fig.6 graphs the new result $b_i = \sqrt{a_i}$ compared with the conventional Eq.(13), $b_i = 2\,a_i - 1$ in terms of *a=1-a*$_i$.

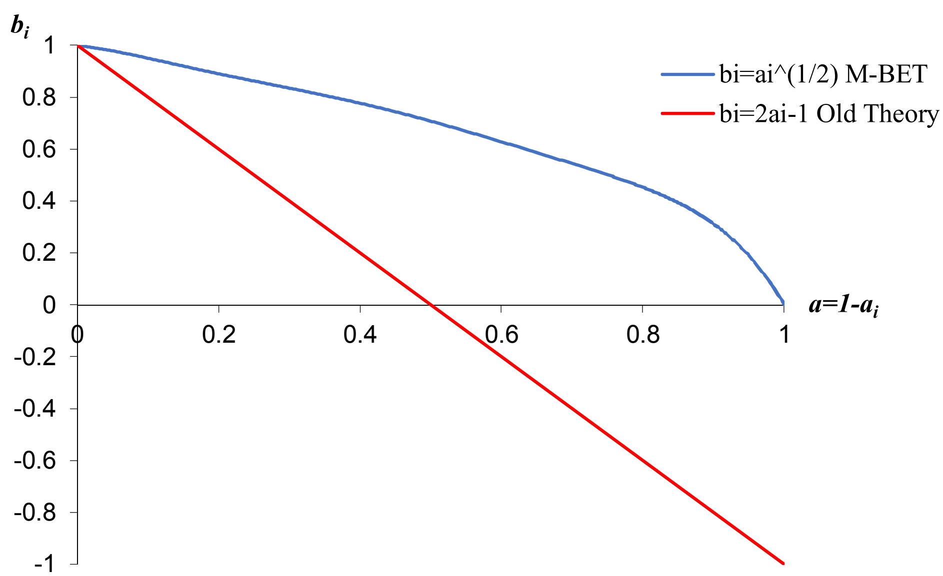

Figure 6 Relationships of *b*$_i$ vs. axial induction, *a=1-a*$_i$

Eq. (28) is the fundamental premise of Mansberger Blade element Theory, M-BET a design algorithm under development by the author. It should be obvious that $b_i = \sqrt{a_i} = \sqrt{(1-a)}$ is a much more realistic solution. The new formula does not break down at axial induction factors above 0.5. Note that conventional theory claims that $b_i < a_i$; from conservation of mass flow the wake therefore would be expanding as the velocity in the wake is decreasing. M-BET claims the opposite that $b_i > a_i$ and the wake therefore must be contracting as its velocity is increasing. Which is correct? Natural observations would agree with the later. I also refer you to not one but two independent sources of empirical data from Doppler radar experiments, Fig. 7 and Fig. 8 from Ref. 7 and Ref. 8 respectfully. The data collected and presented in these papers is truly enlightening. Both sources clearly show that after the initial decrease in velocity through the turbine the velocity increases downwind. Figure 7 clearly shows cross sections of the contracting wake with increasing velocity. Figure 8 graphically depicts the velocity deficit. From figure 8 we can approximate a_i equal to 1 minus the velocity deficit at the turbine or 1-0.36 = 0.64. Similarly b_i= 1 minus the downstream final velocity deficit or 1-0.18 = 0.82. Inserting into Eq. (28) and comparing $b_i = \sqrt{a_i}$, $b_i = \sqrt{0.64} = 0.80$, which is amazingly close and supports the new theory.

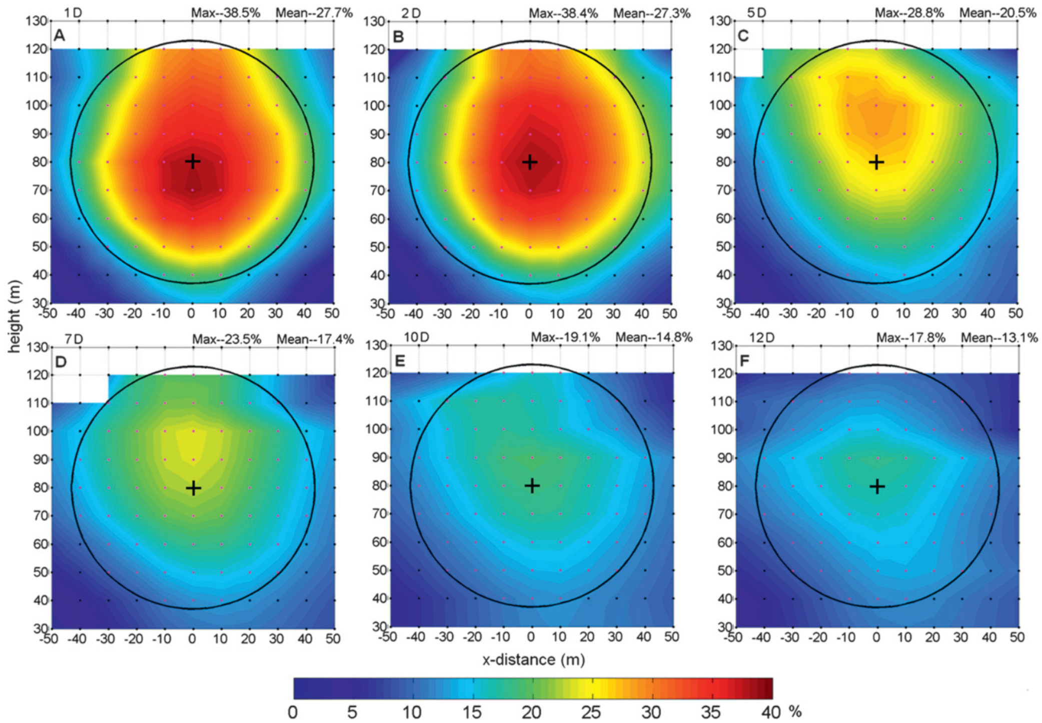

Figure 7 Doppler Radar Data; Hirth B.D. and Schroder J.L.[5]
Clearly Depicted Contracting Wake Diameter and Increasing Velocity

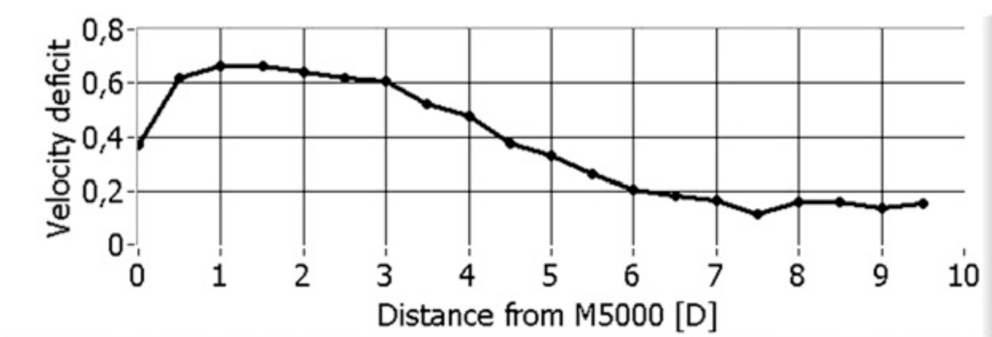

Figure 8 Doppler Radar Data; Kasler, Y., Rahm, S. and Simmet, R.[6]
Decreasing Velocity Deficit Implies Decreasing Wake Diameter and Increasing Velocity

Continuing with the new theory, we can now take our deductive solution Eq. (24) and insert back into Eq. (20) giving us the corrected momentum equation,

$$T = \rho A_6 \left(V_1^2 - V_6^2\right) - \rho V_1^2 A_2 \left(\frac{V_6}{V_1} - 1\right) \tag{29}$$

Dividing through be qA_2 we arrive at the non-dimensional

$$C_T = 2\left[\frac{A_6}{A_2}\left(1 - \frac{V_6^2}{V_1^2}\right) - \left(\frac{V_6}{V_1} - 1\right)\right] \tag{30}$$

From the continuity equation $V_2 A_2 = V_6 A_6$ therefore $\frac{A_6}{A_2} = \frac{V_2}{V_6} = \frac{a_i}{b_i}$ and with $b_i = a_i^{1/2}$ we insert all into Eq. (30) arriving at what I like to call the laminar wake momentum equation.

$$C_T = 2\left(1 - a_i^{3/2}\right) \tag{31}$$

There is additional empirical support for the validity of the laminar wake momentum Eq. (31). Glauert and others, collected and corrected early NACA and BARC wind tunnel data to arrive at the graph I believe was originally depicted by Eggleston and Stoddard[4] in 1987. A version of that graph appears as Fig. (9).

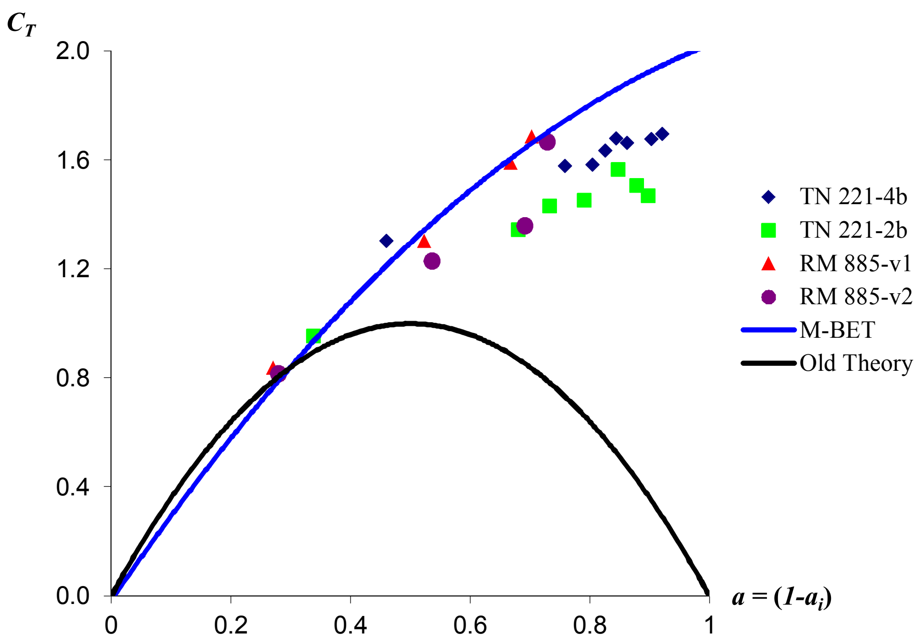

Figure 9 Axial Induction Factor vs. C_T for Empirical Data

Figure 9 clearly shows a far superior fit of the data by the laminar wake momentum equation then by any other historic alternative. Empirical and other new formulas such as the recent technical report by the National Renewable Energy Lab[7] fail to properly correct the relationship between a_i and b_i. Most importantly the Laminar Wake Equation is based on this new relationship and shows great accuracy with the data. Error in some of the data fit may be due to data interpretation, wind tunnel effects or an actual turbulent wake state increasing turbine back pressure shifting results the right.

Continuing with the development of the new theory we solve Eq. (31) for inflow velocity ratio

$$a_i = b_i^2 = \left(1 - 0.5 C_T\right)^{2/3} \tag{32}$$

Equation (32) defines the important inflow velocity ratio $a_i = (1-a)$ the inverse of the axial induction factor in terms of both the final velocity and the thrust coefficient. With this new relationship we can return to the energy equation.

V. The Correct Energy Equation

Now let's look at the true and correct energy equation. It is undisputed in any fluid or thermo-dynamic textbook that if the flow between two regions contains a mechanical device such as a propeller or a wind turbine, Bernoulli's equation is not valid. I found no exceptions to this. Obviously this applies between stations 2-3. As previously stated we also can't analyze between stations 1-2 or 3-6 independently of 2-3 using Bernoulli's equation in the currently accepted manner. Why? Because the most significant term in the energy equation is the internal energy related to ΔT the change in temperature. (Note from here on out the variable T will be used exclusively for temperature and for thrust I will use F_n the force normal to the turbine.) Conventional theory ignores ΔT by assuming temperature before and after the turbine is the same and that the only transfer of energy is from pressure energy independent of temperature. But the true energy equation does not allow the pressure to change independent of temperature.

There are many forms of the general energy equation for dealing with flow through a turbine. For an ideal gas ignoring rotation, the simplest form of the energy equation is

$$c_p T_2 + \frac{1}{2}V_2^2 = c_p T_3 + \frac{1}{2}V_3^2 + e_{out} \tag{33}$$

The term c_p is the specific heat of the fluid at constant temperature. The value of this term for air is often sited at 0.240 Btu/lb°R. This needs to be converted to ft^2/s^2°R for consistent units in the above equation. The conversion is approximately 1 Btu/lb°R = 25,037 ft^2/s^2°R ,therefore the c_p constant at 77°F works out to be equal to approximately 6008. This gives one an idea of the significance of the internal energy of the system versus the kinetic energy. Further notice that according to Eq. (33) if we extract energy from the flow then the velocity cannot remain constant across the turbine without a temperature drop.

For an ideal gas $c_p T$ can be considered equal to enthalpy h, which is tied to pressure and temperature in the following equation

$$c_p T = h = \frac{p}{\rho} + c_v T \tag{34}$$

where c_v also must be converted from its commonly used 0.171 Btu/lb°R to approximately 4,291 ft^2/s^2°R. Equation (34) into Eq. (33) yields

$$\frac{p_1}{\rho_1} + 4291 \cdot T_1 + \frac{1}{2}V_1^2 = \frac{p_3}{\rho_3} + 4291 \cdot T_3 + \frac{1}{2}V_3^2 + e_{out} \tag{35}$$

The above again demonstrates the thermal energy potential in the air flow.

The enthalpy relationship between temperature and pressure can be related by the ideal gas equation of state formula $p = \rho RT$, where R in this case is the gas constant for air equal to 1716.51 ft^2/s^2/°R. Furthermore the relationships between specific heats and the gas constant are related by the following relationships:

$$R = (c_p - c_v) \text{ and } k = {c_p}/{c_v} \text{ ,where k = 1.4 for air.} \tag{36}$$

Using the above, the energy equation between stations 1-6 can be manipulated to look like this,

$$3.5\frac{p_1}{\rho_1} + 0.5V_1^2 = 3.5\frac{p_6}{\rho_6} + 0.5V_6^2 + e_{out} \tag{37}$$

If Eq. (6) and Eq. (7) were in this form a corrected solution for Δp , C_T, and C_p might result, but as can be seen the result is dependent on a change in density and temperature with no straight forward solution, at least not without considering the thermodynamics.

So let's refocus right now on the basic energy equation between stations 1-6 and see what we can deduce. This equation similar to Eq. (33) appears as

$$c_p T_1 + \frac{1}{2}V_1^2 = c_p T_6 + \frac{1}{2}V_6^2 + e_{out} \tag{38}$$

therefore

$$e_{out} = c_p\left(T_1 - T_6\right) + \frac{1}{2}\left(V_1^2 - V_6^2\right) \tag{39}$$

This is energy per unit mass, in order to solve for power out we multiply by mass flow and rearrange terms,

$$P = \rho_2 V_2 A_2 [c_p (T_1 - T_6) + \frac{1}{2}(V_1^2 - V_6^2)] \tag{40}$$

Dividing through by $\frac{1}{2}\rho_1 V_1^3 A_2$, with $\rho_1 \approx \rho_2$ and still governed by the momentum equation result from Eq. (28) $b_i^2 = a_i$ will give us the dimensionless power coefficient,

$$Cp = \frac{2 a_i c_p (T_1 - T_6)}{V_1^2} + a_i - a_i^2 \tag{41}$$

Let us now define the ratio in the change in enthalpy to kinetic energy available as *K*

$$K = \frac{c_p (T_1 - T_6)}{\frac{1}{2}V_1^2} \tag{42}$$

Inserting Eq. (42) into Eq. (41) yields

$$Cp = (K+1)a_i - a_i^2 \tag{43}$$

Equation (43) is the corrected version of Eq. (18) which Betz used to derive his supposed limit. If we take the derivative with respect to a_i, set equal to zero and solve

$$\frac{dCp}{da_i} = K + 1 - 2a_i = 0 \tag{44}$$

$$a_{i,optimum} = \frac{1}{2} + \frac{K}{2} \tag{45}$$

$$Cp_{\max} = \frac{K^2 + 2K + 1}{4} \tag{46}$$

This solution implies the only limiting factor to our power extraction is *K*, our change in enthalpy relationship. If we make the conventional assumption that the temperature is constant then *K*=0 which implies $Cp = a_i - a_i^2$. Taking the derivative of this, setting equal to zero and solving we get a maximum *Cp* of 0.25 at an inflow velocity ratio and or axial induction factor in this case of 0.5. This is far below Betz's supposed limit and occurring at the velocity ratio where conventional theory breaks down. But this is considering the kinetic energy contribution only.

I am not the first to suggest that the kinetic energy contribution could be this low. See references (8) *A Modified Form of the Betz' Wind Turbine Theory Including Losses A.* Dyment 1989, (9) *Reformulation of the Momentum Theory Applied to Wind Turbines* by Ricardo Prado 1995 and (10) *Limits of the Turbine Efficiency for Free Fluid Flow* by Gorbon, Gorlov and Silantyev 2001. These papers independently use revised models of the flow field along with more advanced mathematics to derive maximum power coefficients of closer to 0.30 and less. The results of these papers have apparently been ignored, probably because we know that modern 3 blade wind turbines can achieve power coefficients exceeding 0.45. In retrospect it may become apparent that the fore mentioned authors' results were more correct than the original momentum equation solution when the corrected energy equation is realized.

In order to better understand this, let's assume that a hypothetical wind turbine with a $Cp = 0.40$ is operating at a supposed optimum design axial induction of 1/3 (inflow velocity ratio of 2/3). We can solve for the theoretical ΔT that would be required for the additional performance. Solving Eq. (41) for ΔT

$$\Delta T = \frac{V_1^2}{2c_p a_i}\left(C_P + a_i^2 - a_i\right) \tag{47}$$

Choosing $Cp = 0.40$, $c_p = 6011$, $a_i = 0.67$ and an example wind speed of 30ft/s or roughly 20.5 mph,

$$\Delta T = \frac{30^2}{2(6011)(0.67)}\left(0.40 + 0.67^2 - 0.67\right) = 0.02^\circ F$$

In other words it only takes a temperature drop of $0.02°F$ from the enthalpy term to shift the power coefficient from less than 25% to 40%. If we graph Eq. (43) in the figure below, we can see that the a_i-Cp plane or the result of kinetic energy only, is not representative of the turbine limit but merely the turbine kinetic energy baseline curve.

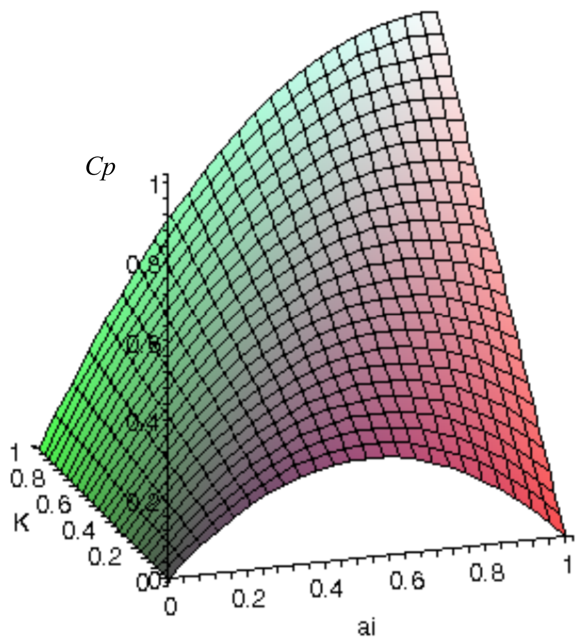

Figure 10 Graph of *Cp* Equation (43)

So we must ask, by what means is this temperature drop and resulting increase in performance accomplished and controlled? In order to understand, we must now put rotational terms into the energy equation. As currently presented the energy equation is still not correct without the critical rotational terms. The power extracted by the wind turbine is equal to

$$P = \tau \cdot \Omega \tag{48}$$

The torque applied to the wind turbine must be equal and opposite of the subsequent torque reacting with the slipstream. The rotation or angular velocity of the slipstream however can and will be different from that of the turbine blades. Therefore we will carefully distinguish between Ω defined as the angular velocity of the turbine from ω, the angular velocity of the slipstream or an annular element of the slipstream. The rotational kinetic energy per unit mass of air contained in an annular element of the slipstream is equal to

$$ke_{\theta r} = \frac{1}{2}\omega^2 r^2 \tag{49}$$

where r is equal to the radial position of the element versus R, which is reserved for the maximum radius of the turbine. The equation for the power output of the wind turbine is derived from Euler's turbine equation which looks like

$$P = \dot{m}\Omega(r_i V_{\theta,i} - r_e V_{\theta,e}) = \dot{m}\,\Omega\, r V_{\theta,e} \tag{50}$$

where V_θ represents the tangential velocity in the direction of rotation which at our turbine inlet is assumed to be zero. At the exit of an annular element V_θ will be equal to ωr. Mass flow, $\dot{m}$ is equal to ρVA, ρ the density of the air, $V=V_2=a_i V_1$ the velocity of the air through the turbine. For the power contained in an infinitesimal annular element, A is the area equal to $2\pi r dr$ at the turbine. We can now put the elemental power equation in the following form

$$dP = 2\pi\rho\; a_i V_1 \Omega\,\omega\, r^3 dr \tag{51}$$

We can derive the rotational energy extracted term by dividing Eq. (51) by $\dot{m} = \rho VA$ the mass flow through the annular element, this yields the energy extracted per unit mass which now appears in terms of the rotational parameters

$$e_{out} = \frac{2\pi\rho\, a_i V_1 \Omega\omega r^3 dr}{(\rho\, a_i V_1 2\pi\, r dr)} = \Omega\omega r^2 \tag{52}$$

We can now insert Eq. (49) and Eq. (52) into the basic energy equation yielding

$$c_p T_1 + \frac{1}{2}V_1^2 = c_p T_6 + \frac{1}{2}V_6^2 + \frac{1}{2}\omega_6^2 r_6^2 + \Omega\omega_3 r_3^2 \tag{53}$$

Equation (53) is the corrected energy equation which is relevant for the design of any horizontal axis wind turbine which extracts energy through rotation. Rearranging and solving for ΔT

$$\Delta T = T_6 - T_1 = \frac{\left[\frac{1}{2}\left(V_1^2 - V_6^2\right) - \Omega\omega_3 r_3^2 - \frac{1}{2}\omega_6^2 r_6^2\right]}{c_p} \tag{54}$$

This answers the earlier question, how do we induce the temperature drop and increase the performance of the turbine. Equation (54) shows the temperature drop must be a function of the rotational parameters alone. As can be seen decreasing the velocity from station 1 to 6 will actually increase the temperature. This therefore implies the performance enhancement of the turbine must come from increasing the rotational term $\Omega\omega_3 r_3^2$.

VI. The Thermodynamic Wind Turbine Model

This theory as with most, assumes a laminar wake unaffected by viscous outer layer or turbulent mixing. In addition it assumes that enough blades are present and rotating at a sufficient velocity that the normal force generated by the turbine blades is distributed evenly throughout the turbine disc area as a pressure distribution. We have corrected both the momentum equation and the energy equation. This basic wake model is drawn based on observations of naturally occurring flow around obstructions and in agreement with the new momentum equation. For the purpose of better understanding the thermodynamics of the airflow through the turbine, let's examine a realistic hypothetical case study for a micro-wind turbine. We will give it the following parameters and conditions;

Diameter 6 ft., R = 3 ft	V_∞ = 30 ft/s
Area = 28.27 ft^2	p_∞ = 2116.2 lb/ft^2
Cp = 0.40	ρ_∞ = 0.002378 slug/ft^3
a=1/3, a_i=2/3	λ_R = 6

I apologize for carrying out this paper and case study in the English system of units; I will revise it to the metric system as soon as time allows.

We will not be using any form of Bernoulli's equation; we will use only the true energy equation, the equation of state $p = \rho RT$ and where appropriate the thermodynamic equations for isentropic flow found in any fluid or thermo-dynamic text as,

$$\left(\frac{T_f}{T_i}\right)^{\left(\frac{k}{k-1}\right)} = \left(\frac{\rho_f}{\rho_i}\right)^{(k)} = \left(\frac{p_f}{p_i}\right) \tag{55}$$

As noted before, very small changes in temperature, hundredths of a degree can have significant effects on the energy equation. Therefore it is essential to carry as many decimals places as possible throughout the calculations and to use correct gas and specific heat constants that obey the relationships in Eq. (36). The following values were found to give reasonable results,

c_p = 6007.79 ft^2/s^2	$R = (c_p - c_v)$ = 1716.51 ft^2/s^2
c_v = 4291.28 ft^2/s^2	$k = c_p / c_v$ = 1.40000

In practice knowing the temperature and pressure, the equation of state would be used to calculate the density. To start the case study the equation of state formula is first used to make sure our initial temperature coincides with our constants.

$$T_1 = \frac{p_1}{\rho_1 R} = \frac{2116.2}{(0.002378)(1716.51)} = 518.440°R$$

A. Free Spinning Turbine

I stated earlier that the thrust or normal force on the turbine times velocity $F_n V$ was not directly associated with power extraction but instead results in a shift in internal energy or enthalpy. I would like to demonstrate this by first analyzing the sample turbine, free spinning with no energy extraction. The total energy in the flow stream is often referred to by the stagnation enthalpy

$$h_0 = c_p T + \frac{1}{2}V^2 = \frac{p}{\rho} + c_v T + \frac{1}{2}V^2 \tag{56}$$

If no energy is extracted this value must remain constant, either way it can be used to solve for other parameters and cross check values with it along the turbine stations. Station 1 is self-explanatory with the given values:

T_1 = 518.440°R
p_1 = 2116.20 lb/ft^2
ρ_1 = 0.002378 slug/ft^3
V_1 = 30 ft/s
h_{01} = 3115128.6 ft^2/s^2

Given the axial induction factor of 1/3, or inflow velocity ratio of 2/3, the velocity at station 2 has been defined as $V_2 = a_i V_1$ = 20 ft/s. Knowing h_0 must remain constant we can use Eq. (56) to solve for T_2 and use Eq. (55) to solve for the rest of the parameters.

T_2 = 518.482°R
p_2 = 2116.7944 lb/ft^2
ρ_2 = 0.00237848 slug/ft^3
V_2 = 20.00 ft/s
h_{02} = 3115128.6 ft^2/s^2

Next, using the laminar wake Eq. (31) we calculate the thrust coefficient.

$$C_T = 2\left(1 - a_i^{3/2}\right) = 2\left(1 - (2/3)^{1.5}\right) = 0.911338$$

The thrust coefficient can be directly correlated to the both the normal force acting on the turbine and the pressure drop between station 2-3 with the following

$$C_T = \frac{F_n}{qA_2} = \frac{-\Delta p_{2,3}}{q} \tag{57}$$

Therefore

$$\Delta p_{2,3} = -qC_T \text{ where } q = \frac{1}{2}\rho V_1^2 \tag{58}$$

For this case

$$q = 0.5(0.002378)(30)^2 = 1.0701 \text{ lb/ft}^2$$

and

$$\Delta p_{2,3} = (1.0701)(-0.911338) = -0.975222 \text{ lb/ft}^2$$

We now have all the information we need to calculate the station 3 parameters. Based on the conservation of mass flow $\dot{m} = \rho VA$ is constant through the turbine. We can make the assumption that $V_2 \approx V_3$ if the change in density is very small and confirm it later. So in order to balance with the momentum equation we know $p_3 = p_2 + \Delta p_{2,3} =$ 2116.2-0.97522= 2115.819 lb/ft^2. In this first analysis we are assuming no power is extracted therefore $T_2 = T_3$, and ρ_3 can be calculated from the equation of state:

<u>Free Spinning</u>
T_3 = 518.4816°R
p_3 = 2115.8192 lb/ft^2
ρ_3 = 0.00237738 slug/ft^3
V_3 = 20.00 ft/s
h_{03} = 3115128.6 ft^2/s^2

Confirming reasonableness, the mass flow, $\dot{m} = \rho_2 V_2 A_2 =$ 1.34478 slug/s. If we solve for $V_3 = \dot{m}/(\rho_3 A_2) =$ 20.009 ft/s, and insert it back in the total energy equation we confirm $V_2 \approx V_3$ and $T_2 \approx T_3$ within 0.0001 degree, insignificant to iterate for more accuracy.

The procedure for solving station 6 is a simple matter of calculating V_6 based on the outflow ratio derived from the momentum equation, $V_6 = b_i V_1 = \sqrt{a_i} V_1$ = 24.4949ft/s. Now plugging V_6 into the total energy Eq. (56) one can solve for T_6. Final pressure p_6 is assumed to be back to atmospheric and final density is calculated from the equation of state.

<u>Free Spinning</u>
T_6 = 518.4650°R
P_6 = 2116.2 lb/ft^2
ρ_6 = 0.00237789 slug/ft^3
V_6 = 24.4949 ft/s
h_{06} = 3115128.6 ft^2/s^2

Notice the final temperature is $0.025°F$ higher. Because no energy was extracted this is the resulting shift from kinetic to internal energy caused by $F_n V$, thrust times velocity, demonstrating, $F_n V$ is in not associated with power extraction other than to deter mass flow.

I use the isentropic relations to work backwards from station 6 to the previous station with the same diameter as A_2 and velocity $\approx V_2$ and called this station 5. Because the velocity is the same $T_5 = T_3$ and rearranging the isentropic equations

$$p_5 = \frac{p_6}{\left(\frac{T_6}{T_5}\right)^{3.5}} = \frac{2116.2}{\left(\frac{518.465}{518.482}\right)^{3.5}} = 2116.438$$

Density is again calculated from the equation of state and all the parameters for station 5 are known.

Free Spinning
$T_5 = 518.4816°R$
$P_5 = 2116.4377$ lb/ft^2
$\rho_5 = 0.00237808$ slug/ft^3
$V_5 = 20.00$ ft/s
$h_{05} = 3115128.6$ ft^2/s^2

Note that although temperature and velocity are the same for stations 3 and 5, pressure and density are not and there is not an isentropic solution from stations 2 to 5 or for any of the stations in between. This region of flow from station 2-5 does not flow in accordance with the isentropic equations, but must still obey the energy and momentum equations. It was strictly energy and momentum equations that solved for station 3. The difference in pressure between 3 and 5 must be reconciled by nature with a change in momentum and area at station 4. This is similar to the reduction in area of a fluid jet as it exits a pressurized reservoir or in this case exits the influence of the turbine. The pressure must increase from p_3 to p_5 passing through p_∞. The velocity must slow to a minimum and then start increasing steadily to V_6. This can conveniently occur if the area expands to a maximum at station 4 coincident with minimum velocity V_4 and p_∞ atmospheric pressure.

I define station 4 as the region of minimum velocity and maximum wake diameter. To solve for station 4 refer to the two control volume diagrams in Figure 11.

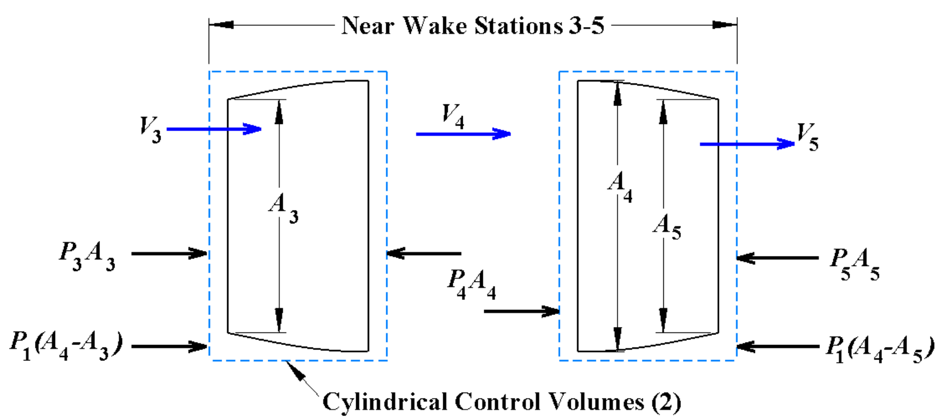

Figure 11 Non Isentropic Wake Region

Assuming the angle of flow to be small and symmetrical on both sides of station 4, then the sum of the forces around each CV diagram must be approximately equal to the change in momentum which yields the following two equations

$$p_3 A_3 + p_\infty (A_4 - A_3) - p_4 A_4 = \dot{m}(V_4 - V_3) \quad (59)$$

$$p_4 A_4 - p_5 A_5 - p_\infty (A_4 - A_5) = \dot{m}(V_5 - V_4) \quad (60)$$

Assuming $p_4 = p_\infty$, and by the definition of station 5 we have $A_3 = A_5 = A_2$ and as well $V_3 \approx V_5 \approx V_2$, then these equations can be simplified to

$$p_3 A_2 - p_\infty A_2 = \dot{m}\,(V_4 - V_2) \tag{61}$$

$$p_\infty A_2 - p_5 A_2 = \dot{m}\,(V_2 - V_4) \tag{62}$$

Now subtracting the bottom equation from the top and solving for V_4 yields

$$V_4 \approx \frac{(p_3 + p_5 - 2p_\infty)A_2}{2\dot{m}} + V_2 \tag{63}$$

This equation (63) reconciles the momentum and energy equations between stations 3-5 and determines the parameters at station 4. Entering values

$$V_4 \approx \frac{(2115.8192 + 2116.4377 - 2(2116.2))28.27}{2(1.3445)} + 20 \approx 18.5 \text{ ft/s}$$

This continued reduction in velocity after leaving the turbine is due to both the continued influence of the turbine as well as the radial momentum of the mass flow. The later may not be adequately accounted for in this theory and the pressure may rebound above p_∞, but Eq.(63) does give us agreement with the characteristic flow. Conservation of mass flow requires an increase in the wake diameter to a maximum at this point. Note, this is only a temporary expansion and is observed in the Doppler radar data graph of Fig. 8.

Continuing with the station 4 solutions, plugging V_4 into the total energy equation one can solve for T_4. Final pressure p_4 is assumed to be passing through atmospheric p_∞ and final density is calculated from the equation of state.

Free Spinning
$T_4 = 518.4864°R$
$P_4 = 2116.20$ lb/ft^2
$\rho_4 = 0.00237779$ slug/ft^3
$V_4 = 18.50$ ft/s
$h_{04} = 3115128.6$ ft^2/s^2

B. Energy Extraction

That completes the solutions for the free spinning turbine. We will now solve for the case with energy being extracted. Solutions for stations 1 and 2 will be identical to those for the free spinning case. Now let's look in depth at the nature of the energy being extracted between stations 2-3. Again I choose a realistic hypothetical $Cp = 0.40$ and $\lambda_R = 6$. By definition of Cp we know the average energy extracted per unit mass flow

$$e_{out} = \frac{C_P q A_2}{\dot{m}} = \frac{C_P 0.5 \rho V_1^3 A_2}{\rho V_2 A_2} = \frac{C_P V_1^2}{2a_i} \tag{64}$$

$$e_{out} = \frac{0.4(30)^2}{2(2/3)} = 270 \text{ ft}^2/\text{s}^2$$

We now need an equation which relates e_{out} and Cp with λ_R and λ_S for a given overall power output. If we integrate either Eq.(51) or (52) from 0 to the outer radius R we arrive at

$$Cp = \frac{\int_0^R 2\pi\rho\, a_i V_1 \Omega\omega r^3 dr}{\int_0^R \rho V_1^3 \pi r dr} = \frac{\frac{1}{2}\pi\rho\, a_i V_1 \Omega\omega R^4}{\frac{1}{2}\rho V_1^3 \pi R^2} = a_i \left(\frac{\Omega R}{V_1}\right)\left(\frac{\omega R}{V_1}\right)$$

$$Cp = a_i \lambda_R \lambda_S \tag{65}$$

Inserting results from Eq. (65) into Eq. (64) yields

$$e_{out} = \frac{1}{2} V_1^2 \lambda_R \lambda_S = \frac{1}{2}\omega\Omega R^2$$
$$\text{or } \lambda_S = \frac{2e_{out}}{\lambda_R V_1^2} \tag{66}$$

We can now examine the relationship between λ_R and λ_S for these conditions

$$\lambda_S = \frac{2e_{out}}{\lambda_R V_1^2} = \frac{2(270)}{6(30)^2} = 0.1$$

or in terms of radial velocity

$$\omega = \frac{V_1 \lambda_S}{R} = \frac{30(.1)}{3} = 1.0 \text{ rad/s} = 9.5 \text{ rpm}$$

compared to

$$\Omega = \frac{V_1 \lambda_R}{R} = \frac{30(6)}{3} = 60 \text{ rad/s} = 573 \text{ rpm}$$

Note the turbine blades are spinning 60 times faster that the rotating slipstream.

We need the above information so we can calculate the rotational kinetic energy per unit mass flow exiting the turbine. This can be derived by integrating Eq. (49) from 0 to R and dividing by total mass flow or by considering the rotational kinetic energy in a solid cylinder of mass flow, either way yielding

$$ke_{\theta R} = \frac{1}{4}\omega^2 R^2 \tag{67}$$

and for our case study

$$ke_{\theta R} = \frac{1}{4}(1)^2 (3)^2 = 2.25 \text{ ft}^2/\text{s}^2$$

Interesting note; comparing this to the rotational energy extracted (2.25/270) = 0.84%, or 1/120th, which implies very little inefficiency at this point in the rotating air mass. A much larger inefficiency occurs in $c_p \Delta T$ = (6007.79)(0.025) = 150 ft^2/s^2 a 56% loss which went into heating the air mass.

We now have the necessary information for the station 3 energy equation with rotational parameters included,

$$h_{02} = c_p T_3 + \frac{1}{2}V_3^2 + \frac{1}{2}\omega\Omega R^2 + \frac{1}{4}\omega^2 R^2 \tag{68}$$

Rearranging to solve for T_3

$$T_3 = \frac{h_{02} - \frac{1}{2}V_3^2 - \frac{1}{2}\omega\Omega R^2 - \frac{1}{4}\omega^2 R^2}{c_p} \tag{69}$$

Inserting our known values with $V_2 \approx V_3$

$$T_3 = \frac{3115128.6 - 0.5(20)^2 - 270 - 2.25}{6007.79} = 518.4363 \text{ °R}$$

Now that we have extracted energy, $h_{02} \neq h_{03}$, but $h_{03} = h_{02} - e_{out}$

$$h_{03} = 3115128.6 - 270 = 3114858.6 \text{ ft}^2/\text{s}^2$$

So the temperature and stagnation enthalpy were determined from the energy equation but the pressure p_3 must still be in balance with the momentum equation. Since we are still assuming the same axial induction factor and thrust coefficient then p_3 must still equal $p_2 + \Delta p_{2,3}$ which equals 2115.819 lb/ft^2. The density is determined from the equation of state fully defining station 3.

Energy Extracted	Free Spinning
T_3 = 518.4363°R	T_3 = 518.4816°R
p_3 = 2115.8192 lb/ft^2	p_3 = 2115.8192 lb/ft^2
ρ_3 = 0.00237759 slug/ft^3	ρ_3 = 0.00237738 slug/ft^3
V_3 = 20.00 ft/s	V_3 = 20.00 ft/s
h_{03} = 3114858.6 ft^2/s^2	h_{03} = 3115128.6 ft^2/s^2

Note in this case we now have a temporary drop in temperature at the turbine of 0.045°F.

The procedure for solving for T_6 is as before using V_6 from the momentum equation in the new total energy equation or

$$T_6 = \frac{h_{03} - \frac{1}{2}V_6^2 - \frac{1}{4}\omega^2 R^2}{c_p} \tag{70}$$

Inserting values into Eq. (77) yields

$$T_6 = \frac{3114858.6 - 0.5(24.4949)^2 - 2.25}{6007.79} = 518.4196$$

Again final pressure p_6 is assumed to be back to atmospheric and final density is calculated from the equation of state.

Energy Extracted	Free Spinning
T_6 = 518.4196°R	T_6 = 518.4650°R
p_6 = 2116.2 lb/ft^2	p_6 = 2116.2 lb/ft^2
ρ_6 = 0.00237809 slug/ft^3	ρ_6 = 0.00237789 slug/ft^3
V_6 = 24.4949 ft/s	V_6 = 24.4949 ft/s
h_{06} = 3114858.6 ft^2/s^2	h_{06} = 3115128.6 ft^2/s^2

Now with energy extracted we have our previously estimated net temperature drop of 0.02°F

Notice that in this analysis of station 6, I do not distinguish a difference between ω_3 or $\omega_{4,5,6}$. An attempt could be made to analyze this difference based on either a balance in pressure with centrifugal forces or conservation of angular momentum and to account for the effect of the theoretical wake diameter on one or the other. But we have succeeded in this theory on estimating the value of ω_3 as being very small and assume any change downwind in rotational kinetic energy to be insignificant in calculating the slipstream parameters and not to have an effect on overall performance in this case. As rotational kinetic energy is increased in new designs this assumption may have to be reevaluated.

Continuing our analysis with station 5 we again solve by working backwards with the isentropic relationships with $V_5 = V_3$ and $T_5 = T_3$.

$$p_5 = \frac{p_6}{\left(\frac{T_6}{T_5}\right)^{3.5}} = \frac{2116.2}{\left(\frac{518.4196}{518.4363}\right)^{3.5}} = 2116.4386$$

Energy Extracted	Free Spinning
T_5 = 518.4363°R	T_5 = 518.4816°R
P_5 = 2116.4386 lb/ft^2	p_5 = 2116.4377 lb/ft^2
ρ_5 = 0.00237829 slug/ft^3	ρ_5 = 0.00237808slug/ft^3
V_5 = 20.00 ft/s	V_5 = 20.00 ft/s
h_{05} = 3114858.6 ft^2/s^2	h_{05} = 3115128.6 ft^2/s^2

We start the station 4 solution with Eq. (63) inserting new values

$$V_4 \approx \frac{(2115.8192 + 2116.4386 - 2(2116.2))28.27}{2(1.3445)} + 20 \approx 18.5 \text{ ft/s}$$

The difference in V_4 between the free spinning versus energy extracting case is noted as insignificant. We can calculate T_4 in the same manner as T_6, $p_4 = p_\infty$, and density is calculated with the equation of state.

Energy Extracted	Free Spinning
T_4 = 518.4411°R	T_4 = 518.4864°R
P_4 = 2116.20 lb/ft^2	p_4 = 2116.2 lb/ft^2
ρ_4 = 0.00237800 slug/ft^3	ρ_4 = 0.00237779 slug/ft^3
V_4 = 18.5 ft/s	V_4 = 18.5 ft/s
h_{04} = 3114858.6 ft^2/s^2	h_{04} = 3115128.6 ft^2/s^2

This completes the analysis of the thermodynamic wind turbine model. Summarizing, it was done in two cases, free spinning and energy extracting to demonstrate that the thrust force alone determines the turbine wake profile and does not contribute to energy extracted. It is shown that a free spinning turbine with a thrust coefficient $C_T = 0.91$ is raising the air temperature by 0.025°F while a similar turbine with the same C_T but extracting power with $C_P = 0.40$ is lowering the temperature by 0.020°F. Implied by this is that turbines operating inefficiently are potentially raising atmospheric temperatures while properly designed efficient wind turbines are actually lower atmospheric temperatures aiding in our fight with global warming. Also a theoretical difference in the near wake and far wake regions is identified. The far wake, like the forward region of turbine influence obeys the isentropic relationships. The near wake is identified as a region of non-isentropic flow. The thermodynamic properties are determined from both the change in momentum due to the thrust and the rotational energy extracted from and imparted into the slipstream. The resulting parameters for the energy extracting case are graphed in the following Fig. 12.

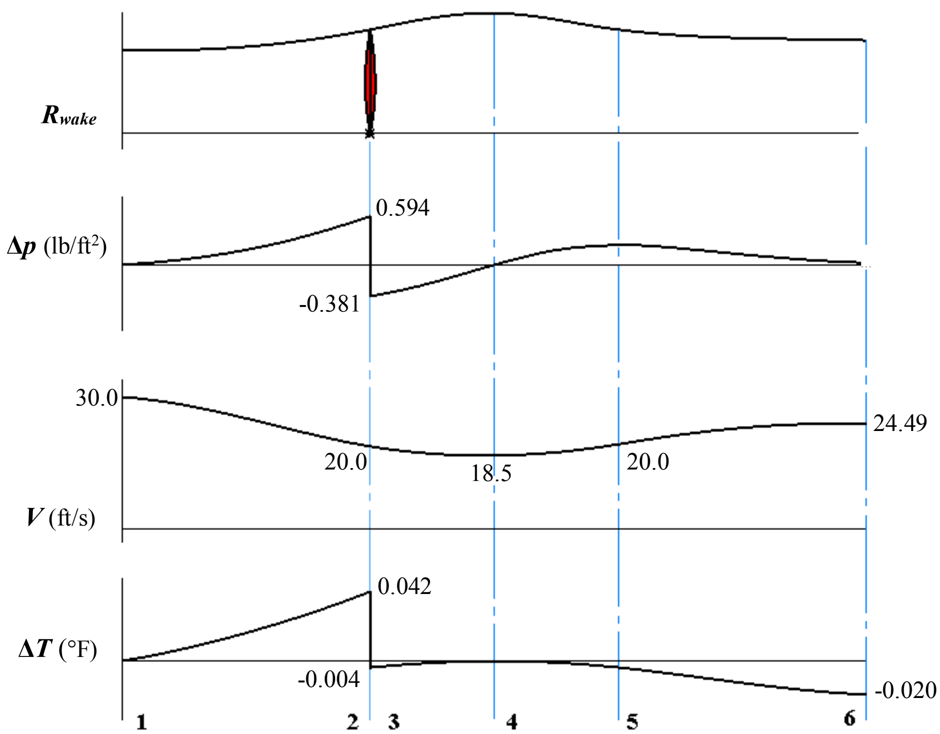

Figure 12 Station Thermodynamic Parameters

VII. Conclusion

This paper has challenged the original derivation of the momentum equation used in wind turbine actuator disc theory along with the energy equation used and the previously mistaken limitations implied by Albert Betz. An alternative solution for the momentum equation has been derived with the fundamental new result that $a_i = b_i^2 = (1 - 0.5C_T)^{2/3}$ and is called the Laminar Wake Momentum Equation. The conventional energy equation for a wind turbine is corrected to include thermodynamic and rotational terms. A mathematical model is presented for a thermodynamically active wind turbine which shows the naturally occurring extraction of thermal energy from any efficiently operated wind turbine.

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